Sets in Excess Demand in Ascending Auctions with Unit-Demand Bidders*

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Abstract

This paper analyzes the problem of selecting a set of items whose prices are to be updated in ascending auctions with unit-demand bidders. A family of sets called "sets in excess demand" is introduced, and it is demonstrated that the selections suggested by Demange, Gale and Sotomayor (J. Polit. Economy 94: 863–872, 1986) and its modification based on the Ford-Fulkerson method proposed by Sankaran (Math. Soc. Sci. 28: 143–150, 1994) belongs to this family. The paper also specifies an ascending auction mechanism where prices for items belonging to a set in excess demand is updated and demonstrates that it converges to the minimum Walrasian price equilibrium.

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Key Words: Multi-item auction; Unit-demand bidders; Excess demand; Algorithms.

1 Introduction

Economies with indivisible items and money have received considerable attention in the literature since the pioneering work of Shapley and Shubik (1972). Shapley and Shubik (1972) did not only prove the existence of a Walrasian equilibrium but also that the set of Walrasian price vectors forms a complete lattice. Consequently, there exist unique minimum and maximum Walrasian equilibrium price vectors. The existence result has later been refined and generalized by e.g. Demange and Gale (1985), Svensson (1983) and Alkan et al. (1991). The lattice property has been demonstrated to play a key role when designing strategy-proof mechanisms. For example, Andersson and Svensson (2008), Demange and Gale (1985) and Leonard (1983) demonstrate that by regarding the minimum Walrasian price equilibrium as a direct mechanism for allocating the indivisible items no agent can gain by strategic misrepresentation.

An obvious field of application for Walrasian pricing mechanisms is auction design. As emphasized by e.g. Ausubel (2004) and Perry and Reny (2005), dynamic auction mechanisms

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are overwhelmingly more prevalent than their direct counterparts (i.e. sealed-bid auctions) because bidders often fear complete revelation of information. Consequently, a variety of different dynamic auction mechanisms has been designed to handle a number of different prerequisites. Many of these mechanisms are only valid under the assumptions that bidders wish to acquire at most one item (unit-demand bidders).[1]

A common ingredient in dynamic auction mechanisms that terminate at the minimum Walrasian equilibrium prices in a unit-demand environment is that prices are updated based on information regarding groups of items that are overdemanded (see e.g. Demange et al., 1986; Sun and Yang, 2009). In this context, a set of items is overdemanded, at a given price vector, if the number of bidders demanding only items in the set is greater than the number of items in the set. This is a natural approach since it is known from a famous theorem by Hall (1935) that a necessary requirement for reaching a Walrasian equilibrium is that all overdemanded sets of items are eliminated. However, it is also well-known that it is in general impossible to reach the minimum Walrasian equilibrium prices by only using information regarding overdemanded sets of items. To see this, suppose that there are three bidders (1, 2 and 3) and two items (A and B) and that bidders 1 and 2 only demand item A and that bidder 3 only demands item B at the current prices. Clearly, the sets $\{A\}$ and $\{A, B\}$ are overdemanded. If, for example, bidder 1 is indifferent between receiving an item or not if the price of item A is raised by one unit, while the other bidders can tolerate higher price increases, then the minimum Walrasian equilibrium prices can never be reached in an ascending process if the prices of both items in $\{A, B\}$ are raised. In fact, the desired equilibrium can be reached only by increasing the price of item A by one unit. This demonstrates that if prices are raised in an arbitrary overdemanded set of items, the minimum Walrasian equilibrium prices need not be reached in the process.

In e.g. Demange et al. (1986) and Sun and Yang (2009), the above problem is solved by restricting the attention to the family of minimal overdemanded sets, consisting of all overdemanded sets with the property that none of its proper subsets is overdemanded (as e.g. the set $\{A\}$ from the above example). In a modification to the auction mechanism in Demange et al. (1986) proposed by Sankaran (1994), a unique overdemanded, but not necessarily minimal overdemanded, set was identified based on the Ford-Fulkerson method and it was demonstrated that the minimum Walrasian price equilibrium will be reached in the process.

As explained above, the price increments cannot be solely based on the overdemand criterion and additional information is needed. This paper considers a subset of the family of overdemanded sets of items consisting of all "sets in excess demand". Formally, an overdemanded set of items S is in excess demand if the number of items in each subset T of S is strictly smaller than the number of bidders that demand some item in T and in addition only demand items in S. Sets in excess demand have a number of attractive properties. First, if a set is minimal overdemanded, at a given price vector, then it is also in excess demand. Hence, the family of minimal overdemanded sets is a subset of the family of sets in excess demand. Second, if two sets are in excess demand, at a given price vector, so is their union. Thus, by taking the union of all sets in excess demand, at given prices, it is possible to construct a unique set in excess demand with a largest cardinality. We demonstrate that the modification to the auction mechanism in Demange et al. (1986) proposed by Sankaran (1994) in fact

[1] Mechanisms that terminate at the minimum Walrasian price equilibrium in a multi-demand setting have for example been considered by Ausubel (2006) and Gul and Stacchetti (2000).

always selects this unique largest set.

Because the selections in both Sankaran (1994) and Demange et al. (1986) are based on sets of items that belong to specific subsets of the family of sets in excess demand, this paper establishes a common framework and a link between Sankaran (1994) and Demange et al. (1986). In addition, the paper also specifies an ascending auction mechanism where in every iteration the prices for items belonging to one of the sets in excess demand is updated. It is demonstrated that this mechanism always converges to the minimum Walrasian price equilibrium. Because it implements the direct mechanism by Leonard (1983), truthful revelation constitutes a Nash equilibrium (see e.g. de Vries et al., 2007).

The paper is organized as follows. Section 2 introduces the economy. Sets in excess demand are defined and characterized in Section 3. Section 4 discusses properties of sets in excess demand when prices are increased and specifies an ascending auction mechanism that always terminates at the minimum Walrasian price equilibrium. Some of the proofs are delegated to the Appendix.

2 The Model

The finite set of bidders is denoted by B and the finite set of items is denoted by I. Each item $i \in I$ has a price p_i and a reservation price r_i. For simplicity and without loss of generality the reservation price of each item is set equal to zero. The value of item $i \in I$ to bidder $b \in B$ is given by the number v_{bi}. These values are assumed to be integers since in reality no bidder can specify a monetary value more closely than to the nearest dollar or cent. For each bidder there is a so-called null-item. All of these items are denoted by 0 and have value normalized to zero, i.e., $v_{b0} = 0$ for all $b \in B$. It is also assumed that a null-item has always price zero, denoted by $p_0 = 0$. Prices are gathered in a vector p. A price vector p is said to be feasible if $p_0 = 0$ and $p_i \geq 0$ for all $i \in I$. For notational simplicity we let $I^* = I \cup \{0\}$ and $I^+(p) = \{i \in I : p_i > 0\}$. The demand correspondence for bidder $b \in B$ at price vector p is defined by:

$$D_b(p) = \{i \in I^* : v_{bi} - p_i \geq v_{bj} - p_j \text{ for all } j \in I^*\}.$$

A feasible price vector p is said to be a Walrasian equilibrium price vector if there is an assignment $x : B \mapsto I^*$ such that $x_b \in D_b(p)$ for all $b \in B$ and if $b' \neq b$ and $x_b = x_{b'}$ then $x_b = 0$, i.e. each bidder is assigned an item from his demand set and if two bidders are assigned the same item then both bidders are assigned the null-item. The pair (p, x) is a Walrasian equilibrium if p is a Walrasian equilibrium price vector and if $x_b \neq i$ for all $b \in B$ then $p_i = 0$, i.e. if an item is not assigned to some bidder, then its price equals the zero reservation price. As demonstrated by Shapley and Shubik (1972) the set of competitive price vectors is non-empty and forms a complete lattice. Thus, the existence of a unique minimum Walrasian equilibrium price vector $p^{\min}$ is guaranteed.

3 Sets in Excess Demand

In this section let p be a given feasible price vector, i.e., $p_0 = 0$ and $p_i \geq 0$ for all $i \in I$. The bidders that *only* demand items in the set $S \subseteq I$ at prices p, and the bidders that demand *some* item in the set $S \subseteq I$ at prices p are collected in the sets

$$O(S,p) = \{b \in B : D_b(p) \subseteq S\},$$
$$U(S,p) = \{b \in B : D_b(p) \cap S \neq \varnothing\},$$

respectively. By definition it holds that $O(S,p) \subseteq U(S,p)$. Consequently, $U(S,p) \cap O(S,p) = O(S,p)$.

Definition 1. A set of items S is overdemanded at prices p if $S \subseteq I$ and $|O(S,p)| > |S|$.

Definition 2. A set of items S is weakly underdemanded at prices p if $S \subseteq I^+(p)$ and $|U(S,p)| \leq |S|$.[2]

Note that Definition 2 requires that S is a subset of $I^+(p)$. Thus, no set S where the price of some item equals the reservation price (zero) can be weakly underdemanded. Consequently, at the reservation prices, no set of items is weakly underdemanded.

Definition 3. A set of items S is in excess demand at prices p if $S \subseteq I$ and:

$$|U(T,p) \cap O(S,p)| > |T| \text{ for each } T \subseteq S. \tag{1}$$

A set of items S is therefore in excess demand if it is overdemanded and the number of items in each proper subset T is strictly smaller than the number of bidders that demand some item in T and in addition only demand items in S. The family of sets in excess demand at prices p is given by:

$$ED(p) = \{S \subseteq I : |U(T,p) \cap O(S,p)| > |T| \text{ for each } T \subseteq S\}.$$

Theorem 1. If $S \in ED(p)$ and $T \in ED(p)$, then $S \cup T \in ED(p)$.

Proof. By Definition 3, we need to prove that the following condition is satisfied:

$$|U(K,p) \cap O(S \cup T,p)| > |K| \text{ for each } K \subseteq S \cup T. \tag{2}$$

Note first that if $b \in O(S,p)$ or $b \in O(T,p)$ then $b \in O(S \cup T,p)$. This together with the observation that there may exist a bidder b with $b \notin O(S,p) \cup O(T,p)$ but $b \in O(S \cup T,p)$ gives: $(U(L,p) \cap O(R,p)) \subseteq (U(L,p) \cap O(S \cup T,p))$ for each $L \subseteq R$ and $R \in \{S,T\}$. Consequently:

$$|U(L,p) \cap O(S \cup T,p)| \geq |U(L,p) \cap O(R,p)| > |L| \text{ for each } L \subseteq R \text{ and } R \in \{S,T\}.$$

The last inequality follows from Definition 3 since $S \in ED(p)$ and $T \in ED(p)$ by assumption. This implies that condition (2) holds if $K \subseteq S$ or $K \subseteq T$.

It remains to prove that condition (2) holds when $K \subseteq S \cup T$ and both $K \nsubseteq S$ and $K \nsubseteq T$. Let $A = K \cap S$ and $C = K \setminus S$, then both A and C are nonempty, $A \cap C = \varnothing$, and $K = A \cup C$. Since $A \subseteq S$ and $C \subseteq T \setminus S$ it holds that:

$$(U(A,p) \cap O(S,p)) \cap (U(C,p) \cap O(T,p)) = \varnothing.$$

[2] This definition is from Mishra and Talman (2010, Definition 3) and it differs slightly from the definition of underdemand in Sotomayor (2002) in the sense that Sotomayor (2002) assumes that there is a dummy bidder that can be allocated more that one item and in addition demands each item at the reservation prices.

Moreover, because $S \in ED(p)$ and $T \in ED(p)$ by assumption, it follows from Definition 3 that both $|U(A,p) \cap O(S,p)| > |A|$ and $|U(C,p) \cap O(T,p)| > |C|$. These facts, together with the observation that there may exist a bidder b with $b \notin O(S,p) \cup O(T,p)$ but $b \in O(S \cup T, p)$, give:

$$\begin{aligned}
|U(K,p) \cap O(S \cup T, p)| &= |U(A \cup C, p) \cap O(S \cup T, p)| \\
&\geq |U(A \cup C, p) \cap (O(S,p) \cup O(T,p))| \\
&\geq |(U(A,p) \cap O(S,p)) \cup (U(C,p) \cap O(T,p))| \\
&= |U(A,p) \cap O(S,p)| + |U(C,p) \cap O(T,p)| \\
&> |A| + |C| \\
&= |A \cup C| \\
&= |K|,
\end{aligned}$$

which concludes the proof. □

Corollary 1. If $ED(p) \neq \varnothing$, then there is a unique set $S^* \in ED(p)$ satisfying $|S^*| > |T|$ for any $T \in ED(p) \setminus \{S^*\}$.

The remaining part of this section demonstrates that (i) the notion of excess demand is a weaker notion than minimal overdemand introduced by Demange et al. (1986), and (ii) that the selection of an overdemanded based on the Ford-Fulkerson method corresponds to the unique set in excess demand with maximum cardinality.

3.1 Minimal overdemand

Definition 4. A set of items S is minimal overdemanded at prices p if S is overdemanded at prices p and:

$$|T| \geq |O(T,p)| \text{ for all } T \subset S.$$

The family of minimal overdemanded sets of items at prices p is given by:

$$MOD(p) = \{S \subseteq I : |O(S,p)| > |S| \text{ and } |T| \geq |O(T,p)| \text{ for all } T \subset S\}.$$

The following result establishes that the notion of sets in excess demand is a weaker notion than minimal overdemand.

Theorem 2. If $S \in MOD(p)$, then $S \in ED(p)$.

Proof. Suppose that $S \in MOD(p)$ and $S \notin ED(p)$. $S \in MOD(p)$ implies that $|O(S,p)| > |S|$ and $|O(T,p)| \leq |T|$ for all $T \subset S$. $S \notin ED(p)$ implies that there exists a non-empty $K \subset S$ such that:

$$|U(K,p) \cap O(S,p)| \leq |K|.$$

Moreover, K is a proper subset of S, since $|O(S,p)| > |S|$ and $U(S,p) \cap O(S,p) = O(S,p)$. Let $S' = S \setminus K$ and note that S' is a non-empty and proper subset of S. It then follows that $O(S',p) = O(S,p) \setminus (U(K,p) \cap O(S,p))$. Thus:

$$\begin{aligned}
|O(S',p)| &= |O(S,p) \setminus (U(K,p) \cap O(S,p))| \\
&\geq |O(S,p)| - |U(K,p) \cap O(S,p)| \\
&> |S| - |K| \\
&= |S'|.
\end{aligned}$$

But then S' is a proper overdemanded subset of S at prices p, which contradicts that $S \in MOD(p)$. Hence, $MOD(p) \subseteq ED(p)$. $\square$

Note that the converse of Theorem 2 is not true, i.e., a set in excess demand need not be a minimal overdemanded set (see e.g. Example 1 in Section 4).

3.2 The Ford-Fulkerson Selection

The Ford and Fulkerson (1956) method is a classical method for network flow problems, that can be used to find a feasible assignment of maximum cardinality. By feasible assignment we mean a set $X(p) \subseteq \{(b,i) : b \in B, i \in I^*\}$ of bidder-item pairs at prices p such that $i \in D_b(p)$ for all $(b,i) \in X(p)$ and such that any $b \in B$ and $i \in I$ is part of at most one pair from $X(p)$. No assumption is made on the cardinality of $X(p)$, but if $|X(p)| = |B|$ each bidder can be assigned an item from his demand set and p is therefore a Walrasian equilibrium price vector.

Starting from $X(p) = \varnothing$, the Ford and Fulkerson (1956) method iteratively updates $X(p)$ based on augmenting paths. An augmenting path with respect to $X(p)$ is a sequence of the form $\mathcal{P} = (b_0, i_0, \ldots, b_n, i_n)$ such that $(b_j, i_{j-1}) \in X(p)$ for $j = 1, 2, \ldots, n$, $i_j \in D_{b_j}(p)$ for $j = 0, 1, \ldots, n$, and $i_j \neq 0$ for $j \neq n$. This characterization corresponds to a bipartite matching problem modified to account for the null-item.

Algorithm 1 (Ford-Fulkerson)**.** Initialize the feasible assignment to the empty set, $X(p) := \varnothing$. For each iteration:

1. Find an augmenting path $\mathcal{P} = (b_0, i_0, \ldots, b_n, i_n)$ with respect to $X(p)$, e.g. using Algorithm 2. If no augmenting path exists, then $X(p)$ is a maximal feasible assignment and terminate the algorithm.

2. Augment the assignment along $\mathcal{P}$:

$$X(p) := (X(p) \setminus \{(b_j, i_{j-1}) : j \in \{1, 2, \ldots, n\}\}) \cup \{(b_j, i_j) : j \in \{0, 1, \ldots, n\}\}.$$

3. Start a new iteration from Step 1.

The augmentation in Step 2 of Algorithm 1 increases the cardinality of $X(p)$ by one. Thus $|X(p)|$ is strictly increasing as long as there is an augmenting path.

Algorithm 2 (Breadth-first Search for an Augmenting Path)**.** Given a feasible assignment $X(p)$, let the initial set of bidders be defined by:

$$B_0 = \{b \in B : (b,i) \notin X(p) \text{ for all } i \in I^*\}, \tag{3}$$

and label any $b \in B_0$ with s. All other bidders and items are initially unlabeled. Introduce the iteration counter n. For each iteration $n = 0, 1, \ldots$:

1. Define the set of items:

$$I_n = \{i \in I^* \setminus \cup_{j=0}^{n-1} I_j : (b,i) \notin X(p) \text{ and } i \in D_b(p) \text{ for some } b \in B_n\}. \tag{4}$$

   Label all $i \in I_n$ with a respective b.

2. If $0 \in I_n$ or there is an $i \in I_n$ such that $(b,i) \notin X(p)$ for all $b \in B$, a shortest augmenting path has been found and terminate the algorithm.

3. If $I_n = \varnothing$, no augmenting path exists and terminate the algorithm.
4. Define the set of bidders:

$$B_{n+1} = \{b \in B \setminus \cup_{j=0}^{n} B_j : (b,i) \in X(p) \text{ for some } i \in I_n\}. \tag{5}$$

   Label all $b \in B_{n+1}$ with a respective i.
5. Set $n := n+1$ and start a new iteration from Step 1.

In case Algorithm 2 terminates with $I_n = \varnothing$, no augmenting path exists and the assignment $X(p)$ is feasible and has maximum cardinality. Otherwise a shortest augmenting path $(b_0, i_0, \ldots, b_n, i_n)$, with $b_j \in B_j$ and $i_j \in I_j$ for $j = 0, 1, \ldots, n$, with respect to $X(p)$ can be found by backtracking the assigned labels starting from 0 if $0 \in I_n$, or from any $i \in I_n$ such that $(b,i) \notin X(p)$ for all $b \in B$, until a bidder with label s is encountered. The breadth-first search is guaranteed to find a shortest augmenting path if one exists. Several possible labels may exist in Steps 1 and 4 of Algorithm 2, and there may be more than one shortest augmenting path. The Ford-Fulkerson method converges regardless of the path and labels chosen.

Theorem 3. Let $S = \cup_{j=0}^{n} I_j$ be the set of labeled items upon termination of Algorithm 2 in the last Ford-Fulkerson iteration at prices p, then S is equal to the unique set S^* in excess demand having maximum cardinality at prices p.

Proof. See the Appendix. □

4 Price Increases and Sets in Excess Demand

This section describes what happens when the prices of items belonging to a set in excess demand are increased. For this purpose, let the family of weakly underdemanded sets of items at prices p be denoted by $WUD(p) = \{S \subseteq I^+(p) : |U(S,p)| \leq |S|\}$.

Lemma 1. Let p be a feasible vector of prices. Suppose that $S \in ED(p)$ and $WUD(p) = \varnothing$ and let:

$$\overline{p}_i = \begin{cases} p_i + 1 & \text{if } i \in S, \\ p_i & \text{otherwise.} \end{cases} \tag{6}$$

Then $WUD(\overline{p}) = \varnothing$.

Proof. We need to show that for an arbitrary $T \subseteq I^+(\overline{p})$ it holds that:

$$|U(T, \overline{p})| > |T|. \tag{7}$$

Three different cases are considered.

Case (i) $T \subseteq S$. We first make three observations. First, $(U(T,\overline{p}) \cap O(S,p)) \subseteq U(T,\overline{p})$ because $O(S,p) \subseteq B$. Second, $D_b(p) \subseteq D_b(\overline{p})$ for all $b \in O(S,p)$ by construction of $\overline{p}$. Consequently, if $b \in U(T,p) \cap O(S,p)$, then $b \in U(T,\overline{p}) \cap O(S,p)$, implying that $|U(T,\overline{p}) \cap O(S,p)| \geq |U(T,p) \cap O(S,p)|$. Third, $|U(T,p) \cap O(S,p))| > |T|$ since $S \in ED(p)$. From these three observations, we conclude:

$$|U(T,\overline{p})| \geq |U(T,\overline{p}) \cap O(S,p)| \geq |U(T,p) \cap O(S,p)| > |T|,$$

which demonstrates that condition (7) is satisfied when $T \subseteq S$.

Case (ii) $T \subseteq I^+(p) \setminus S$. Since $WUD(p) = \varnothing$ and $T \subseteq I^+(p)$ it follows that $|U(T,p)| > |T|$. Moreover, by construction of $\overline{p}$, an item $i \in T$ belongs to $D_b(\overline{p})$ if it belongs to $D_b(p)$, and therefore $U(T,p) \subseteq U(T,\overline{p})$. Hence:

$$|U(T,\overline{p})| \geq |U(T,p)| > |T|,$$

i.e., condition (7) is satisfied when $T \subseteq I^+(p) \setminus S$.

Case (iii) $T = A \cup C$ where $\varnothing \neq A \subseteq S$ and $\varnothing \neq C \subseteq I^+(p) \setminus S$. Clearly, $A \cap C = \varnothing$. By construction of $\overline{p}$ it holds that if $i \in C$ belongs to $D_b(p)$ then it also belongs to $D_b(\overline{p})$. Thus, $U(C,p) \subseteq U(C,\overline{p})$, and as a consequence:

$$|U(T,\overline{p})| = |U(A \cup C,\overline{p})| = |U(A,\overline{p}) \cup U(C,\overline{p})| \geq |U(A,\overline{p}) \cup U(C,p)|. \tag{8}$$

Because, by construction of $\overline{p}$, $\overline{p}_i > p_i$ for $i \in A$ and $\overline{p}_i = p_i$ for $i \in C$, it follows that $U(A,\overline{p}) \cap U(C,p) = \varnothing$, and therefore:

$$|U(A,\overline{p}) \cup U(C,p)| = |U(A,\overline{p})| + |U(C,p)|. \tag{9}$$

Since $A \subseteq S$ it follows from Case (i) that $|U(A,\overline{p})| > |A|$. Moreover, because $WUD(p) = \varnothing$ and $C \subseteq I^+(p)$ it follows that $|U(C,p)| > |C|$. These observations together with (8) and (9) yield:

$$|U(T,\overline{p})| > |A| + |C| = |A \cup C| = |T|,$$

which concludes the proof. □

The main result of this section demonstrates that the following algorithm identifies a minimum Walrasian price equilibrium in a finite number of iterations.

Algorithm 3. Let p^t denote a price vector and set $t := 0$. Initialize the price vector to the reservation prices, $p^0 := r$. For each $t = 0, 1, 2, \ldots$:

1. Collect the demand sets $D_b(p^t)$ of every bidder $b \in B$.
2. If there is no overdemanded set of items at p^t, the algorithm is terminated.
3. Choose a set $S^t \in ED(p^t)$.
4. Compute the updated price vector p^{t+1} whose elements are given by:

$$p_i^{t+1} = \begin{cases} p_i^t + 1 & \text{if } i \in S^t, \\ p_i^t & \text{otherwise.} \end{cases}$$

5. Set $t := t + 1$ and start a new iteration from Step 1.

Theorem 4. Algorithm 3 converges to a minimum Walrasian price equilibrium in a finite number of iterations.

Proof. Let $p^{\min}$ denote the unique minimum Walrasian equilibrium price vector. To prove the theorem, we use a result from Mishra and Talman (2010, Theorem 2) which states that $p = p^{\min}$ if and only if $WUD(p) = \varnothing$ and $OD(p) = \varnothing$. Recall next from Section 3 that $WUD(r) = \varnothing$. From Lemma 1 and the price increases prescribed by Step 4 of Algorithm 3 it follows that $WUD(p^t) = \varnothing$ at any iteration t. Moreover, at any iteration t it holds that $ED(p^t) \neq \varnothing$ as long as there exists at least one overdemanded set of items at p^t. Now the proof follows directly since prices are monotonically increasing and from the observation that as soon as the price of an item becomes sufficiently large, the item cannot belong to any overdemanded set. □

We observe that by Theorem 2 the algorithm described in Demange et al. (1986) is a special case of Algorithm 3, since minimal overdemanded sets always are selected in Step 3 in their algorithm. We also note that the modification of the algorithm in Demange et al. (1986) based on the Ford-Fulkerson method proposed by Sankaran (1994) is a special case of Algorithm 3. This is because the selection in Sankaran (1994) is identical to the set described in Theorem 3.[3]

Example 1. Let $B = \{1, 2, 3, 4, 5\}$ and $I = \{1, 2, 3\}$. The values v_{bi}, $b \in B$, $i \in I$, are given by the matrix:

$$\begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \\ v_{41} & v_{42} & v_{43} \\ v_{51} & v_{52} & v_{53} \end{bmatrix} = \begin{bmatrix} 24 & 8 & 32 \\ 0 & 12 & 66 \\ 99 & 66 & 53 \\ 85 & 30 & 18 \\ 45 & 74 & 94 \end{bmatrix}. \tag{10}$$

For convenience we delete the zero prices for the null-item. At the reservation prices $r = (0, 0, 0)$ the bidders' initial demand sets become $D_1(r) = D_2(r) = D_5(r) = \{3\}$ and $D_3(r) = D_4(r) = \{1\}$. At the reservation prices r we have:

$$ED(r) = \{\{1\}, \{3\}, \{1, 3\}\},$$
$$MOD(r) = \{\{1\}, \{3\}\}.$$

In Figure 1, all possible paths from the seller's reservation prices $r = (0, 0, 0)$ to the minimum Walrasian equilibrium prices $p^{\min} = (79, 46, 66)$ obtainable by Algorithm 3 are represented in the form of a graph.[4] Each vertex corresponds to a price vector reachable by Algorithm 3 and each arc is associated with a specific set in excess demand used to update prices. The solid arcs and the shaded vertices are not reachable using the algorithm in Demange et al. (1986). The unique path trough the graph identified by the algorithm in Sankaran (1994) is the path marked by bold (solid or dashed) arcs to the far right in the figure. □

Appendix: Proof of Theorem 3

Lemma 2. Suppose that Algorithm 2 terminates with $I_n = \varnothing$. Let $S = \cup_{j=0}^{n} I_j$, then $O(S, p) = \cup_{j=0}^{n} B_j$, and for each $i \in S$ there is some $b \in O(S, p)$ such that $(b, i) \in X(p)$.

[3]The algorithm in Sankaran (1994) is terminated if the assignment $X(p)$ of maximal cardinality (identified by Algorithm 1) has the property that $|X(p)| = |B|$. This termination criterium may of course also be used in Algorithm 3.

[4]In the graph a set S^t is fixed until some bidder b with $D_b(p^t) \subseteq S^t$ becomes indifferent to some item $i \notin D_b(p^t)$. In Figure 1, this can be seen e.g. for the set $S^0 = \{3\}$ and the price vector $p^0 = (0, 0, 0)$ where $D_1(p^0) = \{3\}$ but $D_1(\hat{p}) = \{1, 3\}$ for $\hat{p} = (0, 0, 8)$. Such an approach has previously been considered in related problems by e.g. Abdulkadiroğlu et al. (2004) and Andersson and Andersson (2011).

Proof. To show that $O(S,p) = \cup_{j=0}^{n} B_j$ we prove that $b \in \cup_{j=0}^{n} B_j$ if and only if $D_b(p) \subseteq S$. We do this by considering three mutually exclusive cases.

Case (i) $(b,i) \in X(p)$ for some $i \notin S$. Then $D_b(p) \nsubseteq S$, so it must be shown that b is not an element of any B_j. The condition in (5) that $(b',i') \in X(p)$ for some $i' \in I_j$ is always false for $b' = b$ since $i \notin S$. It follows that $b \notin \cup_{j=0}^{n} B_j$.

Case (ii) $(b,i) \in X(p)$ for some $i \in S$. Let $i \in I_m$ for $0 \leq m < n$, where the inequality follows from the assumption $I_n = \varnothing$. Equation (5) gives:

$$B_{m+1} = \{b' \in B \setminus \cup_{j=0}^{m} B_j : (b',i') \in X(p) \text{ for some } i' \in I_m\},$$

i.e. $b \in B_{m+1}$. To show that $D_b(p) \subseteq S$ we need only to compute the set of items I_{m+1} in the next iteration from (4):

$$\begin{aligned} I_{m+1} &= \{i' \in I^* \setminus \cup_{j=0}^{m} I_j : (b',i') \notin X(p) \text{ and } i' \in D_{b'}(p) \text{ for some } b' \in B_{m+1}\} \\ &\supseteq \{i' \in I^* \setminus \cup_{j=0}^{m} I_j : i' \in D_b(p) \setminus \{i\}\} \\ &= D_b(p) \setminus \cup_{j=0}^{m} I_j, \end{aligned}$$

where $i \in I_m$ has been used in the last equality.

Case (iii) $(b,i) \notin X(p)$ for all $i \in I^*$. From (3) we have $b \in B_0$. Insertion in (4) yields $D_b(p) \subseteq I_0$, which concludes the proof of the first statement.

To prove the second statement we observe that $0 \notin S$, and for each $i \in S$ there is some $b \in B$ such that $(b,i) \in X(p)$. Otherwise the algorithm would terminate in Step 2, which violates the assumption $I_n = \varnothing$. From Case (ii) above it follows that $b \in O(S,p)$. □

Lemma 3. Suppose that Algorithm 2 terminates with $I_n = \varnothing$, and let $S = \cup_{j=0}^{n} I_j$. If $b \notin O(S,p)$, then $(b,i) \in X(p)$ for some $i \notin S$.

Proof. The result follows from Lemma 2, and Cases (i) and (iii) from its proof. □

Proof of Theorem 3. To prove the theorem, we demonstrate that if Algorithm 2 terminates with $I_n = \varnothing$, then $S = \cup_{j=0}^{n} I_j$ is the set in excess demand with maximum cardinality at prices p.

By assumption Algorithm 2 is not terminated in Step 2, so it follows that $0 \notin S$, and thus $S \subseteq I$. Lemma 2 states that all items in S are assigned to bidders in $O(S,p)$. It follows directly that $|U(T,p) \cap O(S,p)| \geq |T|$ for all $T \subseteq S$. To prove that $S \in ED(p)$, suppose $|U(T,p) \cap O(S,p)| = |T|$ for some $T \subseteq S$.

Let $K = \{b \in O(S,p) : (b,i) \in X(p) \text{ for some } i \in T\}$ denote the set of bidders in $O(S,p)$ that are assigned an item from T. If $|U(T,p) \cap O(S,p)| = |T|$ then $D_b(p) \cap T = \varnothing$ holds for all $b \in O(S,p) \setminus K$. Using Lemma 2 we have from (4) that $I_n \cap T \neq \varnothing$ only if $B_n \cap K \neq \varnothing$, and from (5) that $B_{n+1} \cap K \neq \varnothing$ only if $I_n \cap T \neq \varnothing$. Since the initial set of bidders satisfies $B_0 \cap K = \varnothing$ we obtain the contradictions $T \nsubseteq S$ and $K \nsubseteq O(S,p)$. Thus $|U(T,p) \cap O(S,p)| > |T|$ for all $T \subseteq S$, which completes the proof that $S \in ED(p)$.

Because S is a set in excess demand we know from Corollary 1 that there exists a unique $S^* \in ED(p)$ of maximum cardinality, and from Theorem 1 it follows that $S \subseteq S^*$. To prove that $S = S^*$, suppose $S \subset S^*$.

Denote the set of bidders that are assigned an item in $S^* \setminus S$ by $L = \{b \in B \setminus O(S,p) : (b,i) \in X(p) \text{ for some } i \in S^* \setminus S\}$, and the set of assigned items in $S^* \setminus S$ by $T = \{i \in S^* \setminus S : (b,i) \in X(p) \text{ for some } b \in B \setminus O(S,p)\}$. Without loss of generality we assume that $T \neq \varnothing$.

Otherwise $O(S^*, p) = O(S, p)$ by Lemma 3, and $U(S^* \setminus S, p) \cap O(S^*, p) = \varnothing$ after applying Lemma 2, thus implying that S^* is not in excess demand.

Note that $D_b(p) \nsubseteq S^*$ may hold for some $b \in L$, and that $U(T, p) \cap O(S^*, p) = \varnothing$ from Lemma 2. We obtain:

$$\begin{aligned}|U(T,p) \cap O(S^*,p)| &= |U(T,p) \cap (O(S^*,p) \setminus O(S,p))| \\ &\leq |U(T,p) \cap L)| = |T|.\end{aligned}$$

Hence, if $S \subset S^*$, then S cannot be in excess demand, thereby proving that S is the set in excess demand with maximum cardinality. □

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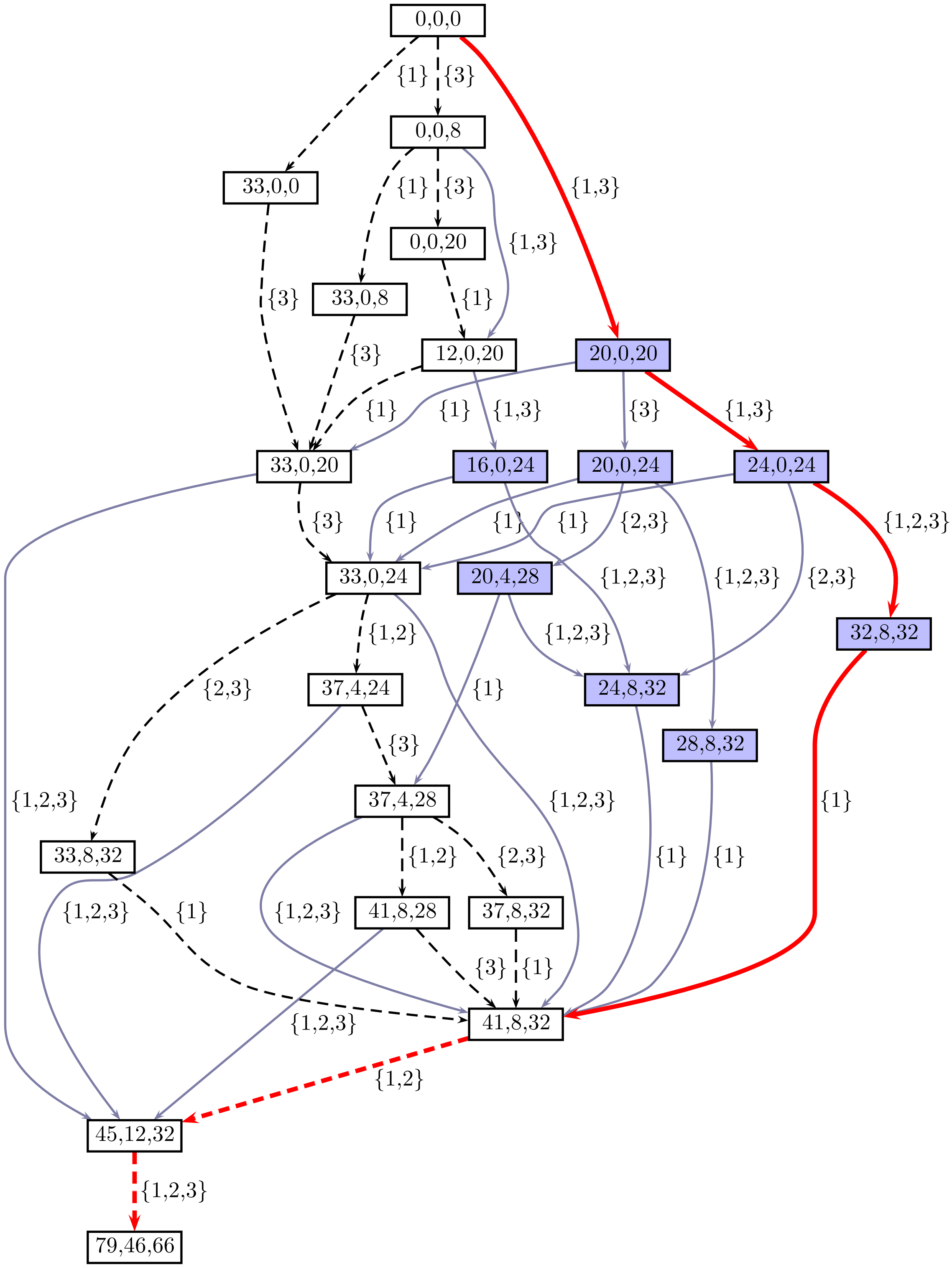

Figure 1: Graph corresponding to an auction with three items and five bidders whose valuations are given by (10). Vertices are labeled with current prices and arcs are labeled with the sets in excess demand used to update prices.

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