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Advanced Microeconomic Analysis – some useful mathematics

Random variables A random variable, X, can take a set of values, x_i, $i = 1, ..., n$ (for a finite random variable generalizations to infinitely many or a continuum of outcomes are obvious but inessential for grasping the concept). Letting , p_i, $i = 1, ..., n$ denote the probabilities, the *expected value* is

$$E(X) = \sum_{i=1}^{n} p_i x_i.$$

Importantly, the expected value is a linear function of a random variable, i.e. for constants a and b,

$$E(aX + b) = aE(X) + b.$$

A few applications:

- *Jensen's inequality.* Let $\overline{x} = E(X)$. For a suitably twice differentiable function, f, (using a Taylor series exansion; see below)

$$f(x_i) = f(\overline{x}) + f'(\overline{x})(x_i - \overline{x}) + f''(\xi_i)\frac{(x_i - \overline{x})^2}{2}, \text{ with } \xi_i \in [\overline{x}, x_i];$$

  now, for a concave function ($f'' < 0$)

$$f(x_i) < f(\overline{x}) + f'(\overline{x})(x_i - \overline{x})$$

  for all i, and taking expectations (using the fact that the expectation is linear[1])

$$E[f(x_i)] < f(\overline{x});$$

  this is Jensen's inequality.

- The *Cumulative Distribution Function* (CDF) of a random variable, X, is defined by

$$F(x) = \Pr\{X \leq x\};$$

  it is a general property of a CDF that it is a non-decreasing function defined on the real line bounded by $0 \leq F(x) \leq 1$.[2]

[1] I.e. $E[f'(\overline{x})(x_i - \overline{x})] = f'(\overline{x})E[(x_i - \overline{x})] = f'(\overline{x})[(\overline{x} - \overline{x})] = 0$.

[2] Depending on the nature of the underlying distribution, a density function, $f(x) = F'(x)$ (continuous distribution) or probability function (discrete distribution) can be derived from the CDF.

Taylor series expansions If a function f is differentiable on an interval $[x_0, x_1]$, $x_0 < x_1$, then by the mean-value theorem,

$$f(x_1) = f(x_0) + f'(\xi)\,(x_1 - x_0)\,, \text{ for some } \xi \in [x_0, x_1];$$

a Taylor series expansion generalizes this to

$$f(x_1) = f(x_0) + f'(x_0)\,(x_1 - x_0) + f''(\xi)\frac{(x_1 - x_0)^2}{2}, \text{ for some } \xi \in [x_0, x_1],$$

and beyond.[3]

Comparative statics – optimization problems Put generally and abstractly, an maximization problem where x is the choice variable and where there is a parameter π that affects the problem. The original objective function can then be expressed $U(x;\pi)$. The first-order condition(s) require that

$$\frac{\partial U(x;\pi)}{\partial x} = 0$$

as an identity (i.e. for all combinations of x and π that solve the equation); this defines x as a function of π, $x(\pi)$, and differentiating with respect to π gives

$$\frac{\partial^2 U(x;\pi)}{\partial x^2}\frac{dx}{d\pi} + \frac{\partial^2 U(x;\pi)}{\partial x \partial \pi} = 0$$

from which

$$\frac{dx}{d\pi} = -\frac{\partial^2 U(x;\pi)/\partial x \partial \pi}{\partial^2 U(x;\pi)/\partial x^2}.$$

Since, moreover, the second-order condition for a maximum is $\partial^2 U(x;\pi)/\partial x^2 < 0$, one can note that generally

$$\text{sign}\,\frac{dx}{d\pi} = \text{sign}\,\frac{\partial^2 U}{\partial x \partial \pi};$$

this is, obviously, reversed for a minimization problem.

[3] That is, ($f^{(k)}$ is the k'th derivative of f)

$$f(x_1) = \sum_{k=0}^{n-1} f^{(k)}(x_0)\frac{(x_1 - x_0)^k}{k!} + f^{(n)}(\xi)\frac{(x_1 - x_0)^n}{n!}.$$