Some notes on integration

By Jerker Holm, September 12, 2010

Consider the area between the *x*-axis and the continuous function $f(x)$ over the interval $[a, x]$, where $f(x) \geq 0$. Let $F(x)$ denote this area. For small changes in *x* the change in the area is given by $F'(x)$. An important and useful observation is that $F'(x) = f(x)$. ($F(x)$ is commonly referred to as the "integral" or the "antiderivative" of $f(x)$.)

Now, consider a function $f(x)$ defined over $[0, \infty]$. Let $F(a)$ and $F(b)$ denote the areas between the *x*-axis and $f(x)$ over the intervals $[0, a]$ and $[0, b]$, respectively. The area under $f(x)$ in the interval $[a, b]$ is then given by $F(b) - F(a)$. This area can also be written with an integral sign as follows:

$$\int_a^b f(x)dx = F(b) - F(a).$$

To find the antiderivative of a function $f(x) = x^a$, the derivation rule is "reversed" so we get $F(x) = \frac{x^{a+1}}{a+1} + (C)$. (The constant, *C*, expresses that antiderivatives with different constants will have the same derivative.)

Example: Find the area under $f(x) = x^2$ in $[0,1] \rightarrow F(x) = \frac{x^3}{3} + C \rightarrow$

$$\int_0^1 x^2 dx = \frac{x^3}{3} + C = F(1) - F(0) = \frac{1}{3}.$$

Some general rules:

$$\int af(x)dx = a\int f(x)dx$$

$$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$$

$$\int [a_1 f_1(x) + a_2 f_2(x) + \ldots + a_n f_n(x)]dx = a_1 \int f_1(x)dx + a_2 \int f_2(x)dx + \ldots + a_n \int f_n(x)dx$$

Applications

Distribution theory: Consider the random variable *X*. The probability that the random variable is equal or less than *a* (a real number) is given by cumulative distribution function $\Pr(X \leq a) = F(a)$, which is directly related to the probability density function $f(x)$ as follows:

$$F(a) = \int_{-\infty}^{a} f(x)dx .$$

Expected utility theory: The return, *R* on an investment is a random variable. The probability that the return is equal or less than *r* is given by $\Pr(R \leq r)$. The outcome for the individual depends on the individual's wealth, *w*, the proportion, α , of wealth invested and the outcome, *r* of the investment. Thus, wealth will be given by $w(\alpha(1+r)+(1-\alpha))$. The resulting wealth will obviously be a random variable and the expected utility will be given by:

$$E[u(w(\alpha(1+r)+(1-\alpha)))] = \int_{-\infty}^{\infty} u(w(\alpha(1+r)+(1-\alpha)))f(r)dr$$